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FOUNDED BY JOHN D. ROCKEFELLER

THE DIFFERENTIAL EQUATIONS OF DYNAMICS

A DISSERTATION
SUBMITTED TO THE FACULTY
OF THE
OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY

BY

ARTHUR CONSTANT LUNN

PRESS OF
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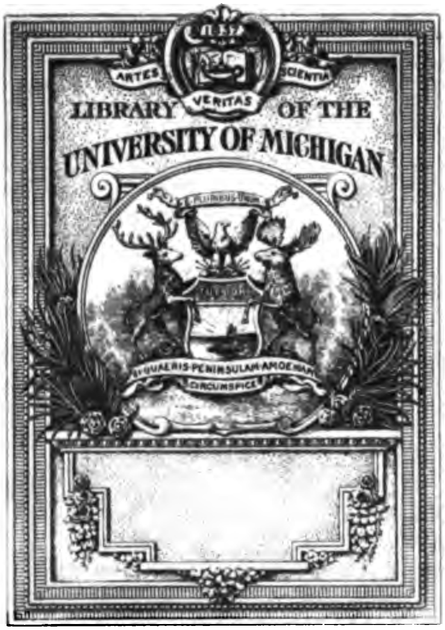
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§ 1. *Introduction.*

The general problem of the dynamics of a system depending on a finite number of parameters may be considered equivalent to the problem offered by the canonical differential equations:

$$(1) \qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad (i = 1 \cdots n),$$

$$F = F(x_1 \cdots x_n,\ y_1 \cdots y_n,\ t),$$

included in the general type of equations in the normal form:

$$(2) \qquad \frac{dx_i}{dt} = X_i(x_1 \cdots x_n,\ t).$$

With respect to the properties of the solutions defined by equations of this type most investigations have naturally referred to the case where the functions X_i, and consequently the solutions x_i, are holomorphic with respect to all arguments, and even where real variables only are primarily considered, admit direct interpretation for complex values. The researches of POINCARÉ in astronomical dynamics,* which have contributed a body of theorems of great importance, have been based mainly on this assumption, though many of his results admit an obvious extension to other cases.

A number of writers have however found it possible, by restricting the variables to real values, to secure a variety of results with more general conditions on the functions X_i. In particular, dynamical applications of the method of real variables are either developed or directly suggested, for example, by papers of POINCARÉ,† BENDIXSON,‡ PICARD,§ PAINLEVÉ,‖ and HADAMARD.¶ In

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* Collected for the most part in *Les Méthodes Nouvelles de la Mécanique Céleste*, Paris, 1892, 1893, 1899.

† *Sur les Courbes définies par les équations différentielles:* Journal de Math., 1881, 1882.

‡ Same title, Acta Math., 24, pp. 1–88.

§ *Sur l'intégration de certaines équations différentielles par les séries:* C. R., 124, pp. 214–217.

‖ *Sur les mouvements et les trajectoires réels des systèmes:* Bull. de la Soc. Math. de France, 22, pp. 136–184.

¶ *Sur certaines propriétés des trajectoires en dynamique:* Jour. de Math., sér. V, 3, pp. 331–387.

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the present paper certain known theorems relative to equations of type (2) are somewhat extended and the results applied to problems of dynamics, as represented by systems of type (1) or by certain auxiliary systems, attention being confined chiefly to the case of real variables, as arguments of functions not in general supposed to be analytic.

The case of chief interest for dynamics, perhaps, is that where the solutions $x_i = \phi_i(t)$ are uniquely determined by the initial conditions $\phi_i(t_0) = x_{i0}$, though sometimes other determining conditions may be important, for instance that a system move from one assigned configuration to another. These solutions, as functions of the initial values x_{i0}, t_0, and of parameters which may appear in the differential equations, have been studied, with special reference to continuity and differentiability, by several writers, in papers which are referred to later.* Some of these results are here deduced as corollaries from a more general theorem on the behavior of the solutions as functions of a parameter taking the values in an arbitrary point-set with a limit-point, a theorem also shown to have as corollary an extension to differential equations of the term-by-term integration of infinite series. This is illustrated by the analysis of resonance in a vibrating system, and by certain methods of approximation in celestial mechanics. In particular there is given a set of conditions sufficient to ensure that the successive approximate solutions furnished by the method of "variation of parameters" shall converge uniformly toward the exact solution during a certain interval of time, together with a proof that in a very general class of cases this method can be defined so that these conditions are satisfied.

§ 2. *The Cauchy-Lipschitz Approximations.*

As basis for the following study of the solutions defined by the differential equations it will be convenient to use the convergent approximations employed in the existence-proof of CAUCHY † and LIPSCHITZ,‡ partly because of the important property to be stated shortly. For the solution $x_1(t), \cdots, x_n(t)$ or $P(t)$, starting, for $t = t_0$, at the point $P_0(x_{10}, \cdots, x_{n0})$, these approximations are obtained by partitioning the interval $t_0 \cdots t$ by numbers in order $t_1, t_2, \cdots, t_s = t$, and successively defining

$$(3) \qquad x_{i,j+1} = x_{ij} + X_i(x_{1j}, \cdots, x_{nj}, t_j)(t_{j+1} - t_j) \quad (i=1\cdots n,\ j=0\cdots s-1).$$

Because of the dependence on the set of numbers marking the partition p, we shall denote such an approximation by $x_i(p, t)$, or $P(p, t)$. Then beside serving to prove in a general case the existence of the solution $P(t)$ for a certain t-interval, these approximations, as shown in papers of PICARD § and PAINLEVÉ,* possess the important property stated in the following lemmas:

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* See notes to § 5.

† MOIGNO: *Leçons de Calcul.* ···, II, p. 385.

‡ *Lehrbuch der Analysis*, II, p. 500.

§ C. R., 128, pp. 1363–1366.

Let D be a closed realm of points $P(x_1, \cdots, x_n)$, and $\bar{D}$ the complementary realm of points $P(\bar{x}_1, \cdots, \bar{x}_n)$, and $t_0 \cdots T$ a closed interval, where T may be greater or less than t_0. Let $X_i(x_1, \cdots, x_n, t)$, $i = 1 \cdots n$, be defined as single-valued functions, for $(x_1, \cdots, x_n)$ any point in D and t any in the interval $t_0 \cdots T$, with the following properties, provided $(x_1', \cdots, x_n')$, $(x_1'', \cdots, x_n'')$ are in D, and t, t', t'' in $t_0 \cdots T$:

A. There are n positive numbers $m_1, \cdots, m_n$, such that

$$|X_i(x_1', \cdots, x_n', t) - X_i(x_1'', \cdots, x_n'', t)| \leqq m_i \sum_{j=1}^{j=n} |x_j' - x_j''|.$$

B. Any positive number ϵ' has a corresponding pair of positive numbers $\theta(\epsilon')$, $\delta(\epsilon')$, such that if $|t' - t''| < \theta(\epsilon')$, and $|x_j' - x_j''| < \delta(\epsilon')$, $j = 1 \cdots n$, then

$$|X_i(x_1', \cdots, x_n', t') - X_i(x_1'', \cdots, x_n'', t'')| < \epsilon'. \tag{5}$$

Let M be the finite upper bound in D and $t_0 \cdots T$ of the moduli $|X_i|$, which exists because of (B): let e, ϵ, be two positive numbers, and $t_0 \cdots t_0 + \tau$ a closed interval, where τ lies between zero and $T - t_0$: define

$$\mu = \sum_1^n m_i, \qquad \epsilon' = \frac{\mu e}{2n(e^{\mu\tau} - 1)}, \qquad I(\tau, \epsilon) = \text{smaller of } \theta(\epsilon'),\ \frac{\delta(\epsilon')}{M}. \tag{6}$$

LEMMA 1. *If* $x_1(t), \cdots, x_n(t)$, *or* $P(t)$, *where* $P(t_0) = P_0$, *exists as a solution of the differential equations* (2), *over the interval* $t_0 \cdots t_0 + \tau$, *and such that for all points of* $\bar{D}$ *and all values of* t *in that interval the lower bound of the sum*

$$\sum_{i=1}^{i=n} |\bar{x}_i - x_i(t)|$$

is not less than $e + \epsilon$: *then for a partition* p *with every interval less than* $I(\tau, \epsilon)$:

$$\sum_{i=1}^{i=n} |x_i(t) - x_i(p, t)| \leqq \epsilon \tag{7}$$

and for any two such partitions p', p'':

$$\sum_{i=1}^{i=n} |x_i(p', t) - x_i(p'', t)| < \epsilon. \tag{8}$$

Conversely, with τ as above, if $P(p, t_0 + \tau)$ exists as such an approximation, with intervals less than $I(\tau, \epsilon)$, and is such that for all the $s + 1$ points

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*C. R., 128, pp. 1505–1508, and Bull. de la Soc. Math. de France, 27, pp. 149–152.

$(x_{1j}, \cdots, x_{nj}), j = 0 \cdots s$, occurring in equations (3), and for every point in $\bar{D}$, the lower bound of the sum

$$\sum_{i=1}^{i=n} | \bar{x}_i - x_i(p, t_j) |$$

is not less than $e + \epsilon$, then for t any in the interval $t_0 \cdots t_0 + \tau$ let p, p', p'' be partitions with intervals less than $I(\tau, \epsilon)$.

Lemma 2. *Under these conditions the partitions p, p', p'' define points in the realm D, and the approximations $P(p', t), P(p'', t)$ satisfy the inequality* (8). *There is a solution $P(t)$ of the differential equations, lying within the realm over the interval $t_0 \cdots t_0 + \tau$, satisfying the inequality* (7), *and such that $P(t_0) = P_0$. For all points of this solution in that interval and all points of $\bar{D}$ the lower bound of the sum*

$$\sum_1^n | \bar{x}_i - x_i(t) |$$

is not less than e.

In particular the conditions of these lemmas are always satisfied if τ, as above, is besides not greater than g/M, and $g + e$ is the lower bound for all points of $\bar{D}$ of the sum

$$\sum_1^n | \bar{x}_i - x_{i0} | .$$

The Cauchy-Lipschitz approximations converge, and the solution is unique, as long as the points $P(t)$ are within the realm D.[*] It is here understood however that the computation can be thought of as begun with intervals of any required fineness.

§ 3. *The solutions as functions of a parameter.*

With the aid of these lemmas the solutions will now be considered as functions of a parameter r, which takes all values in a point-set R admitting r_0 as limit-point. The set R is to contain all its limit-points except r_0, so that the addition of r_0 to the set gives a closed set which may be called R_0. We shall use the term "neighborhood of r_0" to denote all points of R lying within a certain distance of r_0, and denote it by $R(\ )$, with the quantities on which that distance depends indicated within the parenthesis. If R is not bounded infinity may be thought of as one or more definite limit-points, according to convenience in making special interpretations of the results.

Over the realm D and closed interval $T_0 \cdots T$, where $T_0 < T$, let $X_i^r(x_1, \cdots, x_n, t)$, for any value r in R, and $X_i(x_1, \cdots, x_n, t)$, associated with the limit-point r_0, be defined as single-valued functions with the following properties:

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[*] Painlevé, Picard, loc. cit.; see also Lindelöf: Jour. de Math., sér. 5, vol. 6, pp. 423–441.

I. $$|X_i^r(x', t) - X_i^r(x'', t)| \leqq m_i \sum_j |x_j' - x_j''|$$

where m_i is independent of r.

II. The functions X_i^r are continuous in D and $T_0 \cdots T$.

III. For any positive number ϵ there is a neighborhood $R'(\epsilon)$ of r_0, such that

$$|X_i(x, t) - X_i^r(x, t)| < \epsilon$$

if r is in $R'(\epsilon)$. If R has a limit-point r', distinct from r_0, it will also be supposed that the corresponding function satisfies a condition analogous to III, with X_i replaced by $X_i^{r'}$.

The functions X_i are therefore continuous, being uniform limits of continuous functions. Moreover, since the realm D, interval $T_0 \cdots T$ and set R_0 are closed, a modification of the familiar theorem of uniform continuity yields the following

Lemma 3. *For any positive ϵ there are the positive numbers $\delta(\epsilon)$, $\theta(\epsilon)$ depending only on ϵ, and such that if $|x_j' - x_j''| < \delta(\epsilon)$ and $|t' - t''| < \theta(\epsilon)$ then*

$$|X(x', t') - X(x'', t'')| < \epsilon \tag{10}$$

where X is any one of the functions X_i^r, X_i.

For the contrary would imply the existence of an infinite sequence, determining a function, either X_i or an X_i^r, and a point $(x_1, \cdots, x_n, t)$, such that in any neighborhood of that point would be two points for which the values of that function differ by at least ϵ, a violation of continuity. The moduli $|X_i|$, $|X_i^r|$, have thus a single upper bound M.

Now if $(x_1', \cdots, x_n')$, $(x_1'', \cdots, x_n'')$, be two distinct points in D, t a value in $T_0 \cdots T$, and n_i any positive number, there is by III, I, a function X_i^r satisfying the following inequalities:

$$|X_i(x', t) - X_i^r(x', t)| < n_i \sum_j |x_j' - x_j''|,$$

$$|X_i^r(x'', t) - X_i(x'', t)| < n_i \sum_j |x_j' - x_j''|,$$

$$|X_i^r(x', t) - X_i^r(x'', t)| \leqq m_i \sum_j |x_j' - x_j''|,$$

whence by addition:

$$|X_i(x', t) - X_i(x'', t)| < (m_i + 2n_i) \sum_j |x_j' - x_j''|.$$

But n_i is any positive number, so that this is equivalent to

Lemma 4. *The functions X_i also satisfy the condition:*

$$|X_i(x', t) - X_i(x'', t)| \leqq m_i \sum_j |x_j' - x_j''|. \tag{11}$$

It may be noted for example that condition I will be satisfied if the functions

X^r_i, X_i have continuous partial derivatives $\partial X^r_i/\partial x_j$, $\partial X_i/\partial x_j$ satisfying relations analogous to III above.

We now compare the solutions $x_i(t)$ or $P(t)$ of the equations

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n, t) \tag{12'}$$

for the initial conditions $P(t_0) = P_0(x_{10}, \cdots, x_{n0})$, with the solutions $x^r_i(t)$ or $P^r(t)$ of the equations depending on the parameter r:

$$\frac{dx^r_i}{dt} = X^r_i(x^r_1, \cdots, x^r_n, t) \tag{12''}$$

for the initial conditions $P^r(t^r_0) = P^r_0(x^r_{10}, \cdots, x^r_{n0})$; where the initial values x^r_{i0}, t^r_0 also depend on r in such manner that

IV. For any positive numbers e', ϵ', θ', there is a neighborhood $R'(e', \epsilon', \theta')$ of r_0, such that if r is in the set $R'(e', \epsilon', \theta')$, then

$$\sum_j | x^r_{j0} - x_{j0} | < e', \quad | t^r_0 - t_0 | < \theta'. \tag{13}$$

Now let t_0 be interior to the interval $T_0 \cdots T$, τ a positive number not greater than $T - t_0$, and e, ϵ, any two positive numbers.

THEOREM I: *If $P(t)$ is a solution of* (12′) *for all values of t over the interval $t_0 \cdots t_0 + \tau$, and such that for all those values of t, and all points of $\bar{D}$, the lower bound of the sum $\Sigma_i | \bar{x}_i - x_i(t) |$ is not less than $e + \epsilon$; then there exists a neighborhood $R(\epsilon)$ of r_0, such that if r is in the set $R(\epsilon)$, equations* (12″) *have a unique solution $P^r(t)$, determined by the condition $P^r(t^r_0) = P^r_0$, defined over the larger of the intervals $t_0 \cdots t_0 + \tau$ and $t^r_0 \cdots t_0 + \tau$, and such that for every value of t in the common interval*

$$\sum_i | x_i(t) - x^r_i(t) | < \epsilon.$$

The lower bound of the sum $\Sigma_i | \bar{x}_i - x^r_i(t) |$ is not less than e.

Proof. Let the equations (3) be formed with the same partition p for equations (12′) and (12″). To fix the ideas suppose $t^r_0 < t_0$, and the interval $t^r_0 \cdots t_0$ partitioned into k intervals by $t_{-k} = t^r_0, t_{-k+1}, \cdots, t_{-1}, t_0$. Then equations (8) may be written

$$x_{i,j+1} - x_{i,j} = X_i(x_{\sigma,j}, t_j)(t_{j+1} - t_j) \qquad (j = 0 \cdots s-1),$$

$$x^r_{i,j+k+1} - x^r_{i,j+k} = X^r_i(x^r_{\sigma,j+k}, t_j)(t_{j+1} - t_j) \quad (j = -k, \cdots, s-1),$$

whence

$$\begin{aligned}(x_{i,j+1} - x^r_{i,j+k+1}) = (x_{i,j} - x^r_{i,j+k}) &+ \{ X_i(x_{\sigma,j}, t_j) - X^r_i(x_{\sigma,j}, t_j) \} \{ t_{j+1} - t_j \} \\ &+ \{ X^r_i(x_{\sigma,j}, t_j) - X^r_i(x_{\sigma,j+k}, t_j) \} \{ t_{j+1} - t_j \}.\end{aligned}$$

With the abbreviation

$$\sum_i |x_{i,j} - x^r_{i,j+k}| = S^r_j(p), \tag{14}$$

we have therefore by (9), I, and (11):

$$S^r_{j+1}(p) \leqq S^r_j(p) + \{\sum_i |X_i(x_{\sigma,j}, t_j) - X^r_i(x_{\sigma,j}, t_j)| + \mu S^r_j(p)\}\{t_{j+1} - t_j\}.$$

With the positive number λ to be defined put

$$S^r_j(p) + \frac{\lambda}{\mu} = V^r_j(p), \tag{15}$$

then by (9), III, if r is in the set $R'(\lambda/\mu)$:

$$V^r_{j+1}(p) - \frac{\lambda}{\mu} < \left[V^r_j(p) - \frac{\lambda}{\mu} \right] + \left\{ \lambda + \mu \left[V^r_j(p) - \frac{\lambda}{\mu} \right] \right\} \{ t_{j+1} - t_j \}$$

whence

$$V^r_{j+1}(p) < V^r_j(p)\{1 + \mu(t_{j+1} - t_j)\} < V^r_j(p) e^{\mu(t_{j+1} - t_j)}$$

and by multiplication

$$V^r_j(p) < V^r_0(p) e^{\mu(t_j - t_0)} \qquad (j = 1, \cdots, s). \tag{16}$$

But

$$V^r(p) = \frac{\lambda}{\mu} + \sum_i |x_{i,0} - x^r_{i,k}|$$

where

$$\sum_i |x_{i,0} - x^r_{i,k}| \leqq \sum_i |x_{i,0} - x^r_{i,0}| + \sum_i |x^r_{i,k} - x^r_{i,0}|$$

also

$$\sum_i |x^r_{i,k} - x^r_{i,0}| \leqq nM \cdot |t_0 - t^r_0|$$

so that (15), (16) give

$$S^r_j(p) + \frac{\lambda}{\mu} < \left\{ \frac{\lambda}{\mu} + nM \cdot |t_0 - t^r_0| + \sum_i |x_{i,0} - x^r_{i,0}| \right\} e^{\mu(t_j - t_0)}$$

or finally by (14)

$$\sum_{i=1}^{i=n} |x_i(p, t) - x^r_i(p, t)| < \{nM \cdot |t_0 - t^r_0| + \sum_i |x_{i0} - x^r_{i0}|\} e^{\mu\tau_0} + \frac{\lambda}{\mu} \{e^{\mu\tau_0} - 1 \tag{17}$$

where t is any in the interval $t_0 \cdots t_0 + \tau$, and $\tau_0 = T - T_0$. The same form would result for the interval $t^r_0 \cdots t_0 + \tau$ from the supposition $t^r_0 > t_0$.

Put now

$$\begin{aligned} \lambda &= n\epsilon' = \frac{\mu\epsilon}{6\{e^{\mu\tau_0} - 1\}}, \qquad \epsilon' = \frac{\epsilon}{6} e^{-\mu\tau_0} \\ \theta' &= \text{smaller of } \frac{\epsilon}{6nM} e^{-\mu\tau_0},\ t_0 - T; \end{aligned} \tag{18}$$

then if r is in the set $R'(\epsilon', \epsilon', \theta')$ defined by the context of (13):

$$\sum_i |x_i(p, t) - x^r_i(p, t)| < \frac{\epsilon}{2}. \tag{19}$$

Moreover this single partition p, by (10), (11), and Lemma 1, can be taken with intervals less than $I(\tau_0, \epsilon/6)$ defined by (6); in which case

$$\sum_i |x_i(t) - x_i(p, t)| \leqq \frac{\epsilon}{6} \tag{20}$$

and by combination of the last two inequalities:

$$\sum_i |x_i(t) - x_i^r(p, t)| < \frac{2\epsilon}{3}. \tag{21}$$

Inspection of the preceding inequalities shows also that the solution $P(t)$ or the approximation $P^r(p, t)$, according as $t_0 >$ or $< t_0^r$, can be carried back so as to refer to any point in the interval $t_0 - \theta' \cdots t_0 + \tau$, the inequalities (19), (20), (21) remaining fulfilled; and the lower bound of the sum $\Sigma_i |\bar{x}_i - x_i(t)|$ would not be less than $e + \frac{5}{6}\epsilon$. Also the lower bound of the sum $\Sigma_i |\bar{x}_i - x_i^r(p, t)|$ is not less than $e + \frac{1}{6}\epsilon$. By Lemma 2, with ϵ replaced by $\epsilon/6$, the conditions for the existence of the solution $P^r(t)$ are fulfilled for the interval $t_0 - \theta' \cdots t_0 + \tau$, and the theorem is proved, since the combination of inequalities (20), (21), with the inequality

$$\sum_i |x_i^r(t) - x_i^r(p, t)| \leqq \frac{\epsilon}{6} \tag{22}$$

gives

$$\sum_i |x_i(t) - x_i^r(t)| < \epsilon. \tag{23}$$

By a process so closely parallel to the above that it need not be given here can be proved the analogous theorem:

THEOREM II. *If there is a point* r_1 *in the set* $R'(e', \epsilon', \theta')$, *defined according to* (18), *and for some point* $t_0 + \tau$ *in the interval* $t_0 \cdots T$ *a corresponding approximation* $P^r(p, t_0 + \tau)$, *with partition-intervals less than* $I(\tau_0, \epsilon/6)$, *for every point of which and for every point of* $\overline{D}$ *the lower bound of the sum* $\Sigma_i |\bar{x}_i - x_{i,j}^{r_1}|$ *is not less than* $e + \epsilon$; *then if* r *is any point in* $R'(e', \epsilon', \theta')$ *and* p *any partition with intervals less* $I(\tau_0, \epsilon/6)$, *the solutions* $P(t)$ *and* $P^r(t)$ *and the approximations* $P(p, t)$, $P^r(p, t)$ *exist over the interval* $t_0 - \theta' \cdots t_0 + \tau$; *the inequalities* (19) $\cdots$ (23) *are satisfied; and for any of these solutions or approximations the lower bound of* $\Sigma_i |\bar{x}_i - x_i|$ *is not less than* e.

In a similar way the solutions corresponding to two different values of the parameter can be compared with each other instead of with the limiting solution $P(t)$.

The above theorems may be considered a direct extension of the existence theorem of CAUCHY and LIPSCHITZ, in that the existence of a solution for one system of equations is used to show the existence of a solution for another "neighboring" system. Several special forms are at once suggested.

If r is a point $(\mu_1, \cdots, \mu_k)$ of the k-dimensional continuum in the neighborhood of the point r_0, the conditions given are sufficient for the continuity of the solutions as functions of parameters in the ordinary sense, in particular as functions of the initial values x_{i0}, t_0. This case has been treated repeatedly by previous writers, together with the question of the existence of derivatives with respect to the parameters. This latter question will be taken up below in § 5 as a direct application of the preceding theorems. Of more direct interest for the sequel, however, is the application given in the next section to differential equations whose second members are infinite series.

§ 4. *Equations with second members infinite series.*

The following illustration of the results of previous sections gives a direct extension to differential equations of the theorem that a uniformly convergent series of continuous functions can be integrated term-by-term.

Let r_0 be the definite point at infinity, and let the set R include all points $(N_1, \cdots, N_k)$, where the N's are positive integers or zero. Then the X_i^r's may be thought of as defined by the finite series

$$X_i^r = \Sigma X_i^{m_1, \cdots, m_k}(x_1, \cdots, x_n, t) \quad (m_a = 0 \cdots N_a,\ a = 1 \cdots k) \tag{24}$$

and the X_i's by the infinite series

$$X_i = \Sigma X_i^{m_1, \cdots, m_k}(x_1, \cdots, x_n, t) \quad (m_a = 0 \cdots \infty,\ a = 1 \cdots k). \tag{25}$$

Then the conditions imposed in the preceding section are sufficient in order that the solutions corresponding to the finite series shall converge toward those corresponding to the infinite series. An example of considerable generality is given in the following theorem, where for simplicity we suppose $x_{i0}^r = x_{i0}$, $t_0^r = t_0$.

THEOREM III. *If the functions* $X_i^{m_1, \cdots, m_k}(x_1, \cdots, x_n, t)$ *have partial derivatives with respect to* $x_1, \cdots, x_n$, *which are continuous over the realm* D *and interval* $T_0 \cdots T$, *and the infinite series* (25) *and*

$$X_{ij} = \Sigma \frac{\partial}{\partial x_j} X_i^{m_1, \cdots, m_k} \quad (m_a = 0 \cdots \infty,\ a = 1 \cdots k), \tag{26}$$

converge uniformly there; if moreover the equations whose second members are defined by (25) *have a solution* $P(t)$ *lying interior to the realm* D *at every point of the closed interval* $t_0 \cdots t_0 + \tau$, *then there is a point* $(N_1', \cdots, N_k')$ *such that for every point* $(N_1, \cdots, N_k)$, *where* $N_a \geqq N_a'$, *the solutions of the equations*

$$\frac{dx_i^r}{dt} = \Sigma X_i^{m_1, \cdots, m_k}(x_1^r, \cdots, x_n^r, t) \quad (m_a = 0, \cdots, N_a,\ a = 1, \cdots, k), \tag{27}$$

exist and lie interior to the realm D *over the interval* $t_0 \cdots t_0 + \tau$; *and as the* *point* $(N_1, \cdots, N_k)$ *approaches infinity those solutions converge uniformly toward the solution* $P(t)$.

For the functions X_i are then continuous, and have continuous partial derivatives, which are the functions X_{ij} of (27);* the moduli $|X_{ij}|, j = 1 \cdots n$, for the finite and the infinite series have an upper bound m_i, which may be taken as the factor so denoted in (9) I; and since $P(t)$ is interior to the realm D over a closed interval the lower bound of the sum $\Sigma_i |\bar{x}_i - x_i(t)|$ is positive. The conditions of Theorem I are therefore satisfied. It is obvious that the statement of Theorem II, also, presupposing the existence of some $P^r(t)$ instead of $P(t)$, can easily be adapted to this case.

If the interval $t_0 \cdots t_0 + \tau$ is sufficiently short, for example less than g/M, where g is the lower bound, for all points of $\bar{D}$, of $\Sigma_i |x_i - x_{i0}|$, and M is the upper bound of the moduli $|X_i|$, the solutions all lie within D over that interval, and the point $(N'_1, \cdots, N'_k)$ may be taken as $(0, \cdots, 0)$.

A simple case of interest is that of the linear equations:

$$\frac{dy_i}{dt} = \sum_j \phi_{ij}(t) y_j + \psi_i(t), \tag{28}$$

for which the general solution has the form:

$$y_i = \sum_j \theta_{ij}(t) \left\{ y_{j0} + \sum_k \int_{t_0}^{t} \frac{\psi_k \Delta_{kj}}{\Delta} dt \right\}, \tag{29}$$

the functions ϕ_{ij}, ψ_{ij}, being continuous; in which $\theta_{ij}(t_0) = \delta_{ij}$, and Δ is the determinant $|\theta_{ij}|$, $(i, j = 1 \cdots n)$, with minors Δ_{kj} defined by the conditions

$$\sum_j \theta_{kj} \Delta_{lj} = \delta_{kl} \Delta$$

where we put $\delta_{\alpha\beta} = 0$ or 1, as $\alpha \neq \beta$ or $\alpha = \beta$. Then the preceding theorem shows that if the functions ϕ, ψ are uniformly convergent series of continuous functions, the functions θ, their derivatives θ', and the integrals in (29), can be written as series of the same character.

An illustration of this is found in the theory of resonance in vibratory systems, which leads to equations of the type (28), in which the ϕ's are constants, and the ψ's represent the external forces producing the "forced" vibrations. The theorem then gives conditions which justify breaking the particular solutions into series, each term of which corresponds to a component of the disturbing forces. If the external forces can be represented as uniformly convergent series of components which are continuous functions of the time, then each component produces its own distinct effect in the resonance of the system.

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*See e. g.: JORDAN, *Cours d'Analyse*, I, §§ 328, 330.

It is obvious that in the general case of Theorem III the solutions $x_i(t)$ can be thrown into the form of uniformly converging series

$$x_i = \Sigma f_i^{m_1, \dots, m_k}(t) \qquad (m_\alpha = 0 \cdots \infty,\ \alpha = 1 \cdots k) \tag{30}$$

where the terms are defined uniquely by the conditions

$$\sum_{m_\alpha = 0}^{m_\alpha = N_\alpha} f_i^{m_1, \dots, m_k}(t) = x_i^{N_1, \dots, N_\alpha}(t). \tag{31}$$

Combined with Lemma 1, this theorem would yield, for example, the representation of the solutions by infinite series of polynomials in t, as deduced from the Cauchy-Lipschitz series by PICARD* and PAINLEVÉ,† who do not however make clear the fact that the multipliers m_i in Condition I above, or its equivalent, can be considered the same at every step of the process, in other words independent of the parameter r, a property apparently vital to the proof of convergence. That this could not be proved from II, III, and the milder form of I where m_i is replaced by m_i^r, appears from the fact that not every continuous function satisfies Condition I, but every such function is the sum of a uniformly convergent series of polynomials, each of which satisfies that condition. In the case in point, however, Condition I in the form given above is satisfied, provided the polynomial approximations are so chosen that they converge uniformly to the X_i's, and their first partial derivatives converge uniformly to the partial derivatives $\partial X_i / \partial x_j$.

§ 5. *The auxiliary linear equations.*

With the notation of § 3, let $\phi(r)$ be a single-valued function, defined at every point of the set R, and to avoid trivialities let the modulus $|\phi(r)|$ have a lower bound distinct from zero. Then equations (12′), (12″), give:

$$\frac{d\xi_i^r}{dt} = Y_i^r(\xi_1^r, \dots, \xi_n^r, t) \tag{32}$$

where

$$\xi_i^r = \phi(r)\{x_i^r(t) - x_i(t)\}, \tag{33}$$

and the functions Y_i^r are defined by

$$Y_i^r(y_1, \dots, y_n, t) = \phi(r)\left\{X_i^r\left(x + \frac{y}{\phi}, t\right) - X_i(x, t)\right\}. \tag{34}$$

It is understood that $x_1 \cdots x_n$ are thought of as the functions of t defining the solution $P(t)$ of the equations (12′). We now add to the hypotheses of Theorem I the following conditions:

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* C. R., 128, pp. 1363–1366.

† C. R., 128, pp. 1505–1508.

V. For every set of positive numbers ϵ'', e'', θ'', there is a neighborhood $R''(\epsilon'', e'', \theta'')$ of r_0, such that if r is in that subset, then

$$(35) \qquad |\phi(r)\{X_i^r(x, t) - X_i(x, t)\} - X_i'(x, t)| < \epsilon''$$

and

$$(36) \qquad |\phi(r)\{x_{i0}^r - x_{i0}\} - \xi_{i0}'| < e'', \qquad |\phi(r)(t_0^r - t_0) - \eta_0| < \theta''$$

where $X_i'(x, t)$ is a continuous function of $x_1 \cdots x_n$, t over D and $T_0 \cdots T$, and $\xi_{10}', \cdots, \xi_{n0}', \eta_0$ are real numbers.

VI. The functions X_i have partial derivatives

$$\frac{\partial X_i}{\partial x_j} = X_{ij}(x_1, \cdots, x_n, t),$$

continuous over D and $T_0 \cdots T$.

We may therefore define

$$(37) \quad Y_i(y_1, \cdots, y_n, t) = X_i'(x_1, \cdots, x_n, t) + \sum_j X_{ij}(x_1, \cdots, x_n, t) y_j.$$

THEOREM IV. *Under these conditions there is a neighborhood* R_ϵ' *of* r_0, *such that if* r *is in* R_ϵ', *then*

$$\sum_i |\xi_i^r(t) - \xi_i(t)| < \epsilon$$

where the $\xi_i(t)$ *are defined by the equations*

$$(38) \qquad \frac{d\xi_i}{dt} = X_i'(x, t) + \sum_j X_{ij}(x, t)\xi_j$$

with the initial conditions

$$(39) \qquad \xi_{i0} = \xi_i(t_0) = \xi_{i0}' - X_i(x_{10}, \cdots, x_{n0}, t_0)\eta_0.$$

The proof is immediate, since the functions Y_i^r, Y_i, satisfy conditions precisely parallel to I, II, III, above.

Condition I:

$$Y_i^r(\xi'', t) - Y_i^r(\xi', t) = \left\{ X_i^r\left(x + \frac{\xi''}{\phi}, t\right) - X_i^r\left(x + \frac{\xi'}{\phi}, t\right)\right\} \cdot \phi$$

so that

$$|Y_i^r(\xi'', t) - Y_i^r(\xi', t)| \leqq m_i|\phi| \sum_k \left|\frac{\xi_k' - \xi_k''}{\phi}\right| = m_i \sum_k |\xi_k' - \xi_k''|.$$

Condition II: Y_i^r is continuous since X_i^r is continuous, and $\phi(r)$ is nowhere zero.

Condition III: the difference $Y_i^r(\xi, t) - Y_i(\xi, t)$ may be written as the sum of the expressions

$$\phi \cdot \{X_i^r(x + \xi/\phi, t) - X_i(x + \xi/\phi, t)\} - X_i'(x + \xi/\phi, t)$$

and

$$\phi \cdot \{X_i(x + \xi/\phi, t) - X_i(x, t)\} - \sum_j X_{ij}(x, t)\xi_j$$

and

$$X'_i(x + \xi/\phi, t) - X'_i(x, t).$$

The first of these is controlled by V, the second by the theorem of finite differences,* the third by the continuity of X'_i. That the ξ's may be considered as lying in a bounded realm appears if both sides of the inequality (17) are multiplied by $|\phi|$, and the right-hand sides inspected with the conditions imposed in mind. As to the initial conditions we have:

$$\xi_i^r(t_0) = \phi \cdot \{x_i^r(t_0) - x_i(t_0)\} = \phi \cdot \{x_i^r(t_0) - x_i^r(t_0^r)\} + \phi \cdot \{x_i^r(t_0^r) - x_i(t_0)\}$$

or

$$\xi_i^r(t_0) = \phi \cdot (x_{i0}^r - x_{i0}) - \phi \cdot (t_0^r - t_0)[x_i^r(t_0) - x_i^r(t_0^r)]/(t_0 - t_0^r)$$

which may be compared with (36) and (39).

If the set R is the linear continuum, as realm of the single real parameter r, and $\phi(r) = 1/(r - r_0)$, the ξ's are ordinary derivatives, and we recover the results obtained in the memoirs cited in the footnote.† The derivatives $\partial x_i/\partial r$ are defined by the equations

$$\frac{d}{dt}\frac{\partial x_i}{\partial r} = \frac{\partial X_i}{\partial r} + \sum_j X_{ij}\frac{\partial x_j}{\partial r} \tag{40}$$

with the initial conditions

$$\left(\frac{\partial x_i}{\partial r}\right)_{t=t_0} = \frac{\partial x_{i0}}{\partial r} - X_i(x_{10}, \cdots, x_{n0}, t_0)\frac{\partial t_0}{\partial r}. \tag{41}$$

For example, if we put $X_{ij}(x_1, \cdots, x_n, t) = \phi_{ij}(t)$, then the equations

$$y_{ik} = \frac{\partial x_i}{\partial x_{k0}}, \qquad y_{i0} = \frac{\partial x_i}{\partial t_0} \qquad (i, k = 1 \cdots n)$$

define $n + 1$ solutions of the homogeneous linear equations

$$\frac{dy_i}{dt} = \sum_j \phi_{ij}(t) y_i. \tag{42}$$

There must therefore be a linear relation between them. The initial conditions are

$$\left(\frac{\partial x_i}{\partial x_{k0}}\right)_{t=t_0} = \frac{\partial x_{i0}}{\partial x_{k0}} = \delta_{ik}, \qquad \left(\frac{\partial x_i}{\partial t_0}\right)_{t=t_0} = -X_i(x_{10}, \cdots, x_{n0}, t_0). \tag{43}$$

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* JORDAN, *Cours*, I, § 91.

† For continuity and differentiability with respect to initial values and parameters see
PEANO: Torino Atti, 33, pp. 9–18.
NICCOLETTI: Rom. Ac. d. Lincei (5), IV, part 2, pp. 316–324.
ESCHERICH: Wien. Ber., 108, pp. 621–676.
LINDELÖF: J. de Math. (5), 6, pp. 423–441.
SCHUR: Math. Ann., 41, 509.

We find therefore, for $t = t_0$ and consequently for every value of t,

$$\frac{\partial x_i}{\partial t_0} + \sum_k X_k(x_{10}, \cdots, x_{n0}, t_0)\frac{\partial x_i}{\partial x_{k0}} = 0.\text{*} \tag{44}$$

With respect to equation of type (42) it may be noted that if y_i^λ, $\lambda = 1 \cdots k$, be k solutions and we put

$$\xi_{i_1 \ldots i_k} = |y_{i_\mu}^\lambda| \qquad (\lambda, \mu = 1 \cdots k), \tag{45}$$

then

$$\frac{d\xi_{i_1 \ldots i_k}}{dt} = \sum_{a=1}^{a=n} \{\phi_{i_1, a}\xi_{a, i_2, \ldots, i_k} + \phi_{i_2, a}\xi_{i_1, a, i_3, \ldots, i_k} + \cdots + \phi_{i_k, a}\xi_{i_1, \ldots, i_{k-1}, a}\} \tag{46}$$

a system of homogeneous linear equations satisfied by the determinants (45). From the case $k = n$ comes the formula of Liouville:

$$D(t) = D(t_0)e^{\int_{t_0}^{t}(\sum_a \phi_{aa})dt} \tag{47}$$

where D is the determinant $|y_i^j|$, $(i, j = 1, \cdots, n)$. For the system of solutions $\partial x_i/\partial x_{j0}$ we have $D(t_0) = 1$, by (43). This system is therefore a fundamental system.

This shows for instance that if the solutions of (12′) be written

$$x_i = f_i(t, t_0, x_{10}, \cdots, x_{n0})$$

then the equations

$$x_i + \Delta x_i = f_i(t, t_0;\ x_{10} + \Delta x_{10}, \cdots, x_{n0} + \Delta x_{n0}) \tag{48}$$

can be solved to express $\Delta x_{10}, \cdots, \Delta x_{n0}$ uniquely in terms of $\Delta x_1, \cdots, \Delta x_n$, provided the moduli of the latter are properly restricted, since the determinant D is the corresponding Jacobian of the functions f_i, which have moreover continuous partial derivatives with respect to all arguments.†

The same fundamental theorem on implicit functions in real variables shows that the proofs given by POINCARÉ‡ of the existence of families of periodic solutions depending on a parameter apply in part to more general cases than those involving power series.

Suppose the functions $X_i(t, \mu;\ x_1, \cdots, x_n)$ have partial derivatives with respect to $x_1, \cdots, x_n, \mu$, which are continuous functions of all arguments, and that the solutions of the equations

$$\frac{dx_i}{dt} = X_i(t, \mu;\ x_1, \cdots, x_n), \tag{49}$$

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*Compare BENDIXSON, Bull. de la Soc. Math. de Fr., 24, pp. 220–225; LINDELÖF, loc. cit.; HOLMGREN, Stock. Akad. Bih., 25, part 4, p. 19.

†JORDAN, *Cours*, I, § 92.

‡*Méthodes Nouvelles*, I, pp. 83–89.

supposed written in the form

$$x_i = f_i(t, \mu;\ t_0, x_{10}, \cdots, x_{n0}) \tag{50}$$

are such that for a particular set of values $t_0, \mu_0, x_{10}, \cdots, x_{n0}$ the solution has the property

$$x_{i0} = f_i(t_0 + 2\pi, \mu_0;\ t_0, x_{10}, \cdots, x_{n0}) \tag{51}$$

then the equations

$$f_i(t_0 + 2\pi, \mu_0 + \Delta\mu;\ t_0+\Delta t_0, x_{10}+\Delta x_{10}, \cdots, x_{n0}+\Delta x_{n0}) - (x_{i0}+\Delta x_{i0}) = 0, \tag{52}$$

may perhaps be solvable for the increments of the initial values in terms of $\Delta\mu$. Two such cases are obvious from the work of POINCARÉ, without further restrictions on the functions X_i.

First, if the X_i's involve t explicitly, with period 2π, there will exist a family of constant period 2π for all values of μ in the neighborhood of μ_0, if the determinant

$$\left| \frac{\partial x_i}{\partial x_{j0}} - \delta_{ij} \right| \qquad (i, j = 1 \cdots n), \tag{53}$$

is not zero for $t = t_0 + 2\pi$. In this case the condition $\Delta t_0 = 0$ is imposed because of the explicit appearance of t, and $\Delta x_{10} \cdots \Delta x_{n0}$ exist as functions of μ, vanishing with $\Delta\mu$, and determined uniquely, so that the family is simple.

Second, if the X_i's are independent of t explicitly, the period also may be allowed to vary with μ, and this is quite conveniently accomplished by varying the initial instant t_0. In this case if any determinant of the matrix

$$\left| \frac{\partial x_i}{\partial x_{j0}} - \delta_{ij},\ \frac{\partial x_i}{\partial t_0} \right| \qquad (i, j = 1 \cdots n), \tag{54}$$

does not vanish at $t = t_0 + 2\pi$, the increment of the initial constant corresponding to the omitted column may be considered arbitrary, and the other increments are then uniquely defined as functions of that increment and of $\Delta\mu$. For example, the determinant formed by omission of the nth column, on account of the linear relation (44), reduces to

$$\left| \frac{\partial x_i}{\partial x_{j0}} - \delta_{ij},\ X_{i0} \right| \qquad (i = 1 \cdots n, j = 1 \cdots n - 1) \tag{55}$$

where X_{i0} is abbreviation for $X_i(\mu_0; x_{10}, \cdots, x_{n0})$. If t does not enter explicitly in the X_i's, it is in general necessary to allow the period also to vary with μ, since in this case the determinant is always zero, as POINCARÉ has pointed out.* This follows easily from the fact that the functions X_i, considered as functions of t along the initial periodic solution, where $\mu = \mu_0$, will themselves define a periodic solution of equations (42); a case which can happen for homogeneous linear equations with periodic coefficients only in the event that the equation in ω:

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* *Méth. Nouv.*, I, p. 90.

$$(56) \qquad |\theta_{ij}(t_0 + 2\pi) - \delta_{ij}\omega| = 0 \qquad (i, j = 1 \cdots n),$$

has a root equal to unity, the notation being that of (28), (29), with $\psi_i = 0$, so that in the present case θ_{ij} is identical with $\partial x_i / \partial x_{j0}$.

Another illustration of Theorem IV of less obvious character refers to the equations considered in Theorem III. If there are k real positive numbers $\lambda_1, \cdots, \lambda_k$, such that

$$(57) \qquad \lim_{N_a = \infty} N_1^{\lambda_1} \cdots N_k^{\lambda_k} \{X_i - X_i^r\} = 0$$

then similarly for the solutions

$$(58) \qquad \lim_{N_a = 0} N_1^{\lambda_1} \cdots N_k^{\lambda_k} \{x_i(t) - x_i^r(t)\} = 0$$

the ξ's being in this case all equal to zero for $t = t_0$, and consequently for all values of t. Other functions than powers of $N_1, \cdots, N_k$ might enter into similar relations. In particular, if the λ's are all positive and greater than unity, the series defined by (30), (31), are absolutely convergent.

§ 6. *Integrals. Variation of parameters.*

We now consider the initial values $t_0, x_{10}, \cdots, x_{n0}$ as parameters independent of the parameter r, and add to the preceding hypotheses the following, using the notation of § 3.

VII. The functions X_i^r have continuous partial derivatives $\partial X_i^r / \partial x_j$, which satisfy the conditions:

$$(59) \qquad \left| \frac{\partial X_i^r}{\partial x_j} - \frac{\partial X_i}{\partial x_j} \right| < \epsilon$$

if r is in $R'(\epsilon)$.

If a similar condition were imposed with respect to every limit-point of R, this would of course include I above, since the moduli of these derivatives would have an upper bound.

Under these conditions the solutions x_i^r also have partial derivatives with respect to the initial parameters, satisfying the relations:

$$(60) \qquad \frac{\partial x_i^r}{\partial t_0} + \sum_j X_j^r(x_{10}, \cdots, x_{n0}, t_0) \frac{\partial x_i^r}{\partial x_j} = 0$$

parallel to (44).

The existence of these relations has been interpreted by Bendixson and Lindelöf † as the existence of n integrals $\omega_1, \cdots, \omega_n$ of the linear partial differential equation

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* Floquet, J. de l'École Norm., (2), 12, pp. 47–89.

† Loc. cit.

$$\frac{\partial \omega}{\partial t} + \sum_j X_j(x_1, \cdots, x_n, t)\frac{\partial \omega}{\partial x_j} = 0,$$

these integrals being determined by the condition that $\omega_k = x_k$ for $t = t_1$, where t_1 is an assigned value; and the latter writer has specified in a simple way a realm of values $(x_1, \cdots, x_n, t)$, within which those integrals may be known to exist. But for the present purpose it is necessary to inquire as to the dependence of that realm on the parameter r.

If the value of t is considered fixed, the sole condition is that for the initial values $t_0, x_{10}, \cdots, x_{n0}$, the solutions of equations (12′) and (12″) shall all be defined and lie within the realm D, over the closed interval $t_0 \cdots t_1$, and since the moduli $|X_i|$, $|X_i^r|$ have an upper bound independent of r it is possible to indicate such a realm independent of that parameter. The case $t_0 > t_1$ is easily treated in an analogous way.

Let t' be a point within the interval $T_0 \cdots T$ such that the smaller of the numbers $t' - T_0$ and $T - t'$ is θ, and $(x_1', \cdots, x_n')$ a point in D such that the lower bound of the sum $\Sigma_i|\bar{x}_i - x_i'|$ is greater than the positive number δ; and let τ be the smaller of the numbers θ, $\delta/2M$; then the solutions are certainly defined and lie within D over the interval $t_0 \cdots t'$ provided $|t_0 - t'| \leqq \tau$ and $\Sigma_i|x_{i0} - x_i'| \leqq \delta/2$. With a change of notation which will be convenient later we have therefore the following theorem.

Theorem V. *If t' is interior to the interval $T_0 \cdots T$, and $(x_1', \cdots, x_n')$ interior to the realm D, then t' is interior to an interval I within $T_0 \cdots T$, and $(x_1', \cdots, x_n')$ interior to a realm Δ within D, such that for every t in I, every $(x_1, \cdots, x_n)$ in Δ, and every r in R there exist the functions $\omega_1, \cdots, \omega_n$, $\omega_1^r, \cdots, \omega_n^r$, admitting continuous derivatives with respect to the arguments $t, x_1, \cdots, x_n$, and satisfying the equations*

$$\frac{\partial \omega}{\partial t} + \sum_j X_j(x_1, \cdots, x_n, t)\frac{\partial \omega}{\partial x_j} = 0, \tag{61′}$$

$$\frac{\partial \omega^r}{\partial t} + \sum_j X_j^r(x_1, \cdots, x_n, t)\frac{\partial \omega^r}{\partial x_j} = 0, \tag{61″}$$

which are identities with respect to $x_1, \cdots, x_n$ in Δ, and t in I. The moduli of the partial derivatives have an upper bound independent of r, and the moduli of the Jacobians

$$\frac{\partial(\omega_1, \cdots, \omega_n)}{\partial(x_1, \cdots, x_n)}, \qquad \frac{\partial(\omega_1^r, \cdots, \omega_n^r)}{\partial(x_1, \cdots, x_n)},$$

have a lower bound distinct from zero and independent of that parameter.

As chosen above these integrals are uniquely defined by the conditions $\omega_k = \omega_k^r = x_k$ for $t = t'$, and are the solutions of equations (12′), (12″), at the instant t', taking the initial values $x_1, \cdots, x_n$ at the instant t.

It seems needless to make a formal statement of a modified form of this theorem, analogous to Theorem I, whereby the existence of these integrals for equations (61′), for a certain realm Δ and a more extended interval for t than above, might imply the existence of the integrals of (61″) for the same realm and interval, but only for r in a certain subset $R(\epsilon)$. The realm Δ and the set $R(\epsilon)$ could be readily specified by the use of the general inequality (17).

The ω's have the same values for all points of any particular solution, and are of course the integrals or constants of integration, to be identified with the initial constants x_{i0}. To distinguish them from other sets of constants they may be called normal integrals or normal constants. The exact solution $P(t)$ and the approximate solutions $P^r(t)$ are then defined respectively by the equations

$$(62) \qquad \omega_i(x_1, \cdots, x_n, t) = x_{i0}, \qquad \omega_i^r(x_1^r, \cdots, x_n^r, t) = x_{i0},$$

where again for simplicity we suppose the initial constants independent of r.

The results above show therefore that the equations

$$\omega_i(x_1 + \Delta x_1, \cdots, x_n + \Delta x_n, t) = x_{i0} + \Delta x_{i0},$$

$$\omega_i^r(x_i^r + \Delta x_i^r, \cdots, x_n^r + \Delta x_n^r, t) = x_{i0} + \Delta x_{i0},$$

can be solved so as to give the departures Δx_i, Δx_i^r, of the solutions for any value t, in terms of the departures Δx_{i0} of the initial constants, for solutions in the neighborhood of the assigned solution; but it must be remembered that the restrictions on the magnitude of the increments Δx_{i0} under which that solving can be known to be possible depend in general on the values of t and r, and it is conceivable that there should be no single neighborhood of the point $(x_{10}, \cdots, x_{n0})$ for which those restrictions are fulfilled simultaneously for all values of t and r. Inspection of the proof of the theorem on implicit functions above quoted shows, however, that those limitations on the increments are defined by certain inequalities to be satisfied by quotients of determinants, the denominators being the Jacobians just mentioned. Now for the interval of t and the range of initial values limited as specified in obtaining Theorem V, the moduli of the partial derivatives and of the reciprocals of the Jacobians have upper bounds independent of t and r. It is therefore possible to define the inequalities in question by the use of a set of positive numbers independent of t and r. Similar conclusions hold if the solutions $P(t)$ and $P^r(t)$ "cross" at any value in the interval named.

For the following we require to consider the transformations of variables from $(x_1, \cdots, x_n, t)$ to $(\omega_1, \cdots, \omega_n, t)$ or $(\omega_1^r, \cdots, \omega_n^r, t)$, defined by the equations

$$(63) \qquad \omega_i(x_1, \cdots, x_n, t) = \omega_i, \qquad \omega_i^r(x_1, \cdots, x_n, t) = \omega_i^r.$$

It is then shown by the foregoing that if $(x_{10}, \cdots, x_{n0})$ is interior to D and t to the interval $T_0 \cdots T$, then there is a closed set Δ_0 of points $(x_{10} + \Delta x_{10}, \cdots, x_{n0} + \Delta x_{n0})$

and a closed interval I_0 or $t_0 - \theta \cdots t_0 + \theta$, such that for any point on the solution $P(t)$ which starts in Δ_0 for $t = t_0$, or any point on a solution $P^r(t)$ which crosses such a solution, the transformations are uniquely defined and have unique inverses. For example $(x_1, \cdots, x_n)$ at t determines $(x_{10}, \cdots, x_{n0})$ or $(\omega_1, \cdots, \omega_n)$ which determines $(x_1^r, \cdots, x_n^r)$ at t, which determines $(x_{10}^r, \cdots, x_{n0}^r)$ or $(\omega_1^r, \cdots, \omega_n^r)$. *There is a common realm within which there is a one-to-one correspondence between the sets of variables* $(x_1, \cdots, x_n, t)$, $(\omega_1, \cdots, \omega_n, t)$, $(\omega_1^r, \cdots, \omega_n^r, t)$. This realm contains in particular as interior points all points on the assigned solution $P(t)$ for a certain t-interval.

The method of variation of parameters may be defined as the employment of a transformation of variables such that the new variables are integrals or constants of integration for an approximate system of differential equations. If the process employed contemplates only a finite number of such transformations there is no question of convergence to be considered. But with respect to dynamical problems, like the problem of n bodies, this method, unless combined with others, may yield the required simplicity and symmetry of treatment at each stage only by involving an infinite sequence of transformations. In this case also it may be considered that one obtains an approximate solution by treating the variables at a certain stage of the process as constants, and then the question arises whether these approximate solutions converge toward the exact solution.

The preceding theorems furnish means of testing this convergence for certain general classes of differential equations, for example where the approximating equations are defined by series as in § 4. For the determination of the exact solution for a certain interval by an infinite sequence of transformations, however, one more condition must be satisfied; *the approximate solutions must be determined by the method for a certain common interval.* As applied to transformations this would imply the condition that all the transformations employed must have a common realm of definition in the space where the coördinates are $x_1 \cdots x_n$, that realm having all points of the required solution as interior points. Under the hypotheses of the beginning of this section this condition is satisfied for the variation of parameters, if at each stage the integrals chosen are the normal integrals.

THEOREM VI. *For differential equations of the type defined in theorem* III *there is a certain interval for* t *over which the solution starting at an interior point of* D *for* $t = t_0$ *is the uniform limit of the approximate solutions furnished by the method of variation of parameters, if the integrals at each stage are chosen as the normal integrals.*

It is not essential that all approximate solutions have the same initial values x_{i0} as the required solution, but simply such as lie in the neighborhood of that point and converge toward it. It is clear that many other choices of integrals might be made so as to satisfy the required conditions.

§ 7. *Holomorphic functions.*

Attention has thus far been confined to real variables. It is known, however, that the method of difference-equations employed in the proofs applies equally well to analytic functions for complex values of the arguments, so that the results given above can be adapted to such cases also, with the interval $t_0 \cdots t_0 + \tau$ replaced by a certain path of integration in the complex plane, leading to the point at which the value of the solution is sought. If the path of integration be defined by a real parameter it is sufficient to consider equations where the dependent variables, parameters, and functional values are complex, but the independent variable real.

For equations of the type

$$\frac{dz_i}{dt} = Z_i(z_1, \cdots, z_n, \mu, t) \tag{64}$$

the linear equations of § 5 furnish a simple means of verifying the theorem of POINCARÉ relative to the analytic character of the solutions as functions of their initial values, or of the parameter μ. Let the functions Z_i be assumed complex, and holomorphic with respect to the complex arguments $z_1, \cdots, z_n, \mu$, but simply continuous with regard to the real variable t.

If the initial values $z_{10}, \cdots, z_{n0}$ *are holomorphic with respect to* μ, *the solutions are.* In particular the solutions are holomorphic with respect to the initial values.*

For, let

$$Z_i = X_i + \sqrt{-1}\,Y_i, \qquad z_i = x_i + \sqrt{-1}\,y_i, \qquad \mu = \rho + \sqrt{-1}\,\sigma,$$

then equations (64) may be replaced by

$$\frac{dx_i}{dt} = X_i(x, y, \rho, \sigma, t), \qquad \frac{dy_i}{dt} = Y_i(x, y, \rho, \sigma, t), \tag{65}$$

which are equations in real variables, satisfying the conditions of § 5, if the initial values lie in the realm of regularity of the functions Z_i. The derivatives $\partial x_i/\partial\rho, \partial x_i/\partial\sigma, \partial y_i/\partial\rho, \partial y_i/\partial\sigma$, therefore exist and are defined by the equations

$$\frac{d}{dt}\frac{\partial x_i}{\partial\rho} = \sum_j\left(\frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial\rho} + \frac{\partial X_i}{\partial y_j}\frac{\partial y_j}{\partial\rho}\right) + \frac{\partial X_i}{\partial\rho},$$

$$\frac{d}{dt}\frac{\partial x_i}{\partial\sigma} = \sum_j\left(\frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial\sigma} + \frac{\partial X_i}{\partial y_j}\frac{\partial y_j}{\partial\sigma}\right) + \frac{\partial X_i}{\partial\sigma},$$

$$\frac{d}{dt}\frac{\partial y_i}{\partial\rho} = \sum_j\left(\frac{\partial Y_i}{\partial x_j}\frac{\partial x_j}{\partial\rho} + \frac{\partial Y_i}{\partial y_j}\frac{\partial y_j}{\partial\rho}\right) + \frac{\partial Y_i}{\partial\rho},$$

$$\frac{d}{dt}\frac{\partial y_i}{\partial\sigma} = \sum_j\left(\frac{\partial Y_i}{\partial x_j}\frac{\partial x_j}{\partial\sigma} + \frac{\partial Y_i}{\partial y_j}\frac{\partial y_j}{\partial\sigma}\right) + \frac{\partial Y_i}{\partial\sigma}.$$

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* *Méth. Nouv.*, I, p. 58.

By virtue of the conditions of monogeneity:

$$(66)\qquad \frac{\partial X_i}{\partial x_j}-\frac{\partial Y_i}{\partial y_j}=\frac{\partial X_i}{\partial y_j}+\frac{\partial Y_i}{\partial x_j}=\frac{\partial X_i}{\partial \rho}-\frac{\partial Y_i}{\partial \sigma}=\frac{\partial X_i}{\partial \sigma}+\frac{\partial Y_i}{\partial \rho}=0$$

We have therefore

$$(67)\qquad \begin{aligned} \frac{d}{dt}\left(\frac{\partial x_i}{\partial \rho}-\frac{\partial y_i}{\partial \sigma}\right)&=\sum_j\left\{\frac{\partial X_i}{\partial x_j}\left(\frac{\partial x_j}{\partial \rho}-\frac{\partial y_j}{\partial \sigma}\right)+\frac{\partial X_i}{\partial y_j}\left(\frac{\partial x_j}{\partial \sigma}+\frac{\partial y_j}{\partial \rho}\right)\right\},\\ \frac{d}{dt}\left(\frac{\partial x_i}{\partial \sigma}+\frac{\partial y_i}{\partial \rho}\right)&=\sum_j\left\{-\frac{\partial X_i}{\partial y_j}\left(\frac{\partial x_j}{\partial \rho}-\frac{\partial y_j}{\partial \sigma}\right)+\frac{\partial X_i}{\partial x_j}\left(\frac{\partial x_j}{\partial \sigma}+\frac{\partial y_j}{\partial \rho}\right)\right\}, \end{aligned}$$

a system of homogeneous linear equations satisfied by the $2n$ quantities

$$\frac{\partial x_i}{\partial \rho}-\frac{\partial y_i}{\partial \sigma},\qquad \frac{\partial x_i}{\partial \sigma}+\frac{\partial y_i}{\partial \rho}.$$

Because of the conditions on the initial values

$$(68)\qquad \frac{\partial x_{i0}}{\partial \rho}-\frac{\partial y_{i0}}{\partial \sigma}=\frac{\partial x_{i0}}{\partial \sigma}+\frac{\partial y_{i0}}{\partial \rho}=0,$$

we have therefore for every value of t:

$$(69)\qquad \frac{\partial x_i}{\partial \rho}-\frac{\partial y_i}{\partial \sigma}=\frac{\partial x_i}{\partial \sigma}+\frac{\partial y_i}{\partial \rho}=0.$$

A minimum value for the radius of convergence of the resulting power-series in μ can be given by reference to the general inequalities of § 3.

§ 8. *The canonical equations.*

We now consider briefly the special forms assumed by some of the preceding results for the canonical equations (1). The conditions imposed on the functions X_i, X_i^r, will be satisfied if F, F^r, for all values of r in R have all first and second derivatives with respect to $x_1, \cdots, x_n, y_1, \cdots, y_n$, these derivatives being continuous functions of all arguments, including r in the sense that they satisfy the appropriate convergence conditions analogous to III of § 3, with respect to every limit-point of R.

The linear equations of § 5 take the form*

$$(71)\qquad \begin{aligned} \frac{d\xi_i}{dt}&=\sum_j\left(\frac{\partial^2 F}{\partial y_i\partial x_j}\xi_j+\frac{\partial^2 F}{\partial y_i\partial y_j}\eta_j\right)+\left\{\frac{\partial^2 F}{\partial y_i\partial \mu}\right\},\\ \frac{d\eta_i}{dt}&=\sum_j\left(-\frac{\partial^2 F}{\partial x_i\partial x_j}\xi_j-\frac{\partial^2 F}{\partial x_i\partial y_j}\eta_j\right)-\left\{\frac{\partial^2 F}{\partial x_i\partial \mu}\right\}, \end{aligned}$$

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* POINCARÉ, *Méth. Nouv.*, I, p. 166.

which are also canonical, since they can be written

$$\frac{d\xi_i}{dt}=\frac{\partial\Phi}{\partial\eta_i},\qquad \frac{d\eta_i}{dt}=-\frac{\partial\Phi}{\partial\xi_i}. \tag{72}$$

If we put

$$\Phi=\sum_{ij}\left[\tfrac{1}{2}\frac{\partial^2F}{\partial x_i\partial x_j}\xi_i\xi_j+\frac{\partial^2F}{\partial x_i\partial y_j}\xi_i\eta_j+\tfrac{1}{2}\frac{\partial^2F}{\partial y_i\partial y_j}\eta_i\eta_j\right]$$
$$+\sum_j\left\{\frac{\partial^2F}{\partial x_j\partial\mu}\xi_j+\frac{\partial^2F}{\partial y_j\partial\mu}\eta_j\right\}, \tag{73}$$

and ξ, η are derivatives with respect to μ. For derivatives with respect to the initial values the terms in $\{\ \}$ are to be omitted.

If the system in complex variables (64) is canonical, the corresponding real system is also canonical. For if we put

$$x_i=x_i'+\sqrt{-1}x_i'',\qquad y_i=y_i'+\sqrt{-1}y_i'',\qquad F=F'+\sqrt{-1}F'',$$

equations (1) take the form

$$\frac{dx_i'}{dt}=\frac{\partial F'}{\partial y_i'},\qquad \frac{dy_i'}{dt}=-\frac{\partial F'}{\partial x_i'},\qquad \frac{dy_i''}{dt}=\frac{\partial F'}{\partial x_i''},\qquad \frac{dx_i''}{dt}=-\frac{\partial F'}{\partial y_i''}, \tag{74}$$

by reason of the conditions of monogeneity.

In employing transformations of variables it is convenient to choose such systems as retain the canonical form of the equations. It appears from the work of JACOBI* that if a new set of variables $\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n,t$, is defined by the solution of the equations

$$y_i=\frac{\partial S}{\partial x_i},\qquad \alpha_i=\frac{\partial S}{\partial\beta_i} \tag{75}$$

where $S(x_1,\cdots,x_n,\beta_1,\cdots,\beta_n,t)$ is an arbitrary function, then the transformed differential equations are

$$\frac{d\alpha_i}{dt}=\frac{\partial}{\partial\beta_i}\left(F+\frac{\partial S}{\partial t}\right),\qquad \frac{d\beta_i}{dt}=-\frac{\partial}{\partial\alpha_i}\left(F+\frac{\partial S}{\partial t}\right). \tag{76}$$

For instance, if S satisfies the partial differential equation

$$\frac{\partial S}{\partial t}+F_1\left(x_1,\cdots,x_n,\frac{\partial S}{\partial x_1},\cdots,\frac{\partial S}{\partial x_n},t\right)=0 \tag{77}$$

and F is the sum of F_1 and F_2, then equations (76) are

$$\frac{d\alpha_i}{dt}=\frac{\partial F_2}{\partial\beta_i},\qquad \frac{d\beta_i}{dt}=-\frac{\partial F_2}{\partial\alpha_i} \tag{78}$$

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* *Vorlesungen über Dynamik*, lectures 20, 36.